Functional renormalization group approach to composite-particle excitations

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July 10, 2014 @ Mathematics Days in Sofia

Introduction

Why quantum field theories are important in physics?

Particle physics: QFT can

1. combine “special relativity” and “quantum mechanics”, and
2. explain so many experimental results with Standard Model.

Condensed matter physics: QFT can

1. be closely related to statistical physics, and
2. provide convenient description of many-body quantum mechanics.

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Important!

These propositions are closely related with one another.

Role of renormalization group in QFT

Particle physics: What happens if we divide matters? Is there minimal unit?
⇒ Short range, or high frequency, phenomena are their main concern.

Condensed matter physics: What happens if matters accumulate?

1. More is different! (P. W. Anderson, 1972)
2. Long range phenomena are their concern.

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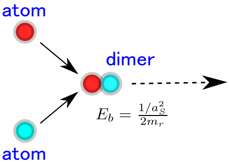
Renormalization Group
= Technology connecting physical phenomena connecting different scales.

Purpose of this talk

- Review on functional renormalization group (FRG) methods
- Application of FRG to compute a many-body state of composite particles.



Condensation of tightly-bound dimers

Functional renormalization group

QFT: What's that? How can we treat it?

QFT: A set of correlation functions satisfying "axioms".

Generating functional of connected Green functions:

$$\exp(W[J]) = \int \prod_x \mathrm{d}\Phi(x) \exp\left(-S[\Phi(x)] + \sum_x J(x)\cdot\Phi(x)\right).$$

Continuum limit? Borel summability of perturbation theories?

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Possible remedies:

1. Lattice field theory & Monte Carlo simulation
2. Construct nonperturbative relations of Green functions!
(⇒ Functional techniques)
    - Dyson-Schwinger equations

$$\int \prod_y \mathrm{d}\Phi(y) \frac{\partial}{\partial\Phi(x)} \exp\left(-S + J\cdot\Phi\right) = 0.$$

    - Functional renormalization group (FRG)

Wilsonian renormalization group

Classical action/Hamiltonian: $S[\phi]$.

"Block spins": $\phi_p = \int \mathrm{d}^d x e^{-\mathrm{i}px}\phi(x)$ for small momenta p.

Field theoretic formulation of RG (Wilson, Kogut 1974, Wegner, Houghton, 1972):

$$\exp -S_\Lambda[\phi] := \mathcal{N}_\Lambda \int \prod_{|p|\geq\Lambda} \mathrm{d}\phi_p \exp -S[\phi].$$

Important!

Correlation functions $\langle \phi_{p_1}\cdots\phi_{p_n}\rangle$ with low momenta $|p_i|\leq\Lambda$ are calculable with S_Λ instead of the microscopic action S.

Functional RG (I)

Schwinger functional W_k with an IR regulator R_k:

$$\exp(W_k[J]) = \int \mathcal{D}\phi \exp\left(-S[\phi] - \frac{1}{2}\phi\cdot R_k\cdot\phi + J\cdot\phi\right).$$

R_k: IR regulator, which controls low-energy excitations ($p^2 \le k^2$).

k-derivatives of the both sides:

$$\begin{aligned}\partial_k \exp(W_k[J]) &= \int \mathcal{D}\phi - \frac{1}{2}\phi\partial_k R_k\phi \exp\left(-S[\phi] - \frac{1}{2}\phi R_k\phi + J\phi\right)\\ &= -\frac{1}{2}\frac{\delta}{\delta J}\partial_k R_k \frac{\delta}{\delta J}\exp W_k[J]\end{aligned}$$

Flow equation (Polchinski, 1984, ...)

$$\partial_k W_k = \int \mathrm{d}x\mathrm{d}y\left(-\frac{1}{2}\frac{\delta W_k}{\delta J(x)}\partial_k R_k(x,y)\frac{\delta W_k}{\delta J(y)} - \frac{1}{2}\partial_k R_k(x,y)\frac{\delta^2 W_k}{\delta J(x)\delta J(y)}\right).$$

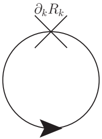
Functional RG (II)

The 1PI effective action $\Gamma_k$is introduced via the Legendre trans.:

$$\Gamma_k[\varphi] + \frac{1}{2}\varphi \cdot R_k \cdot \varphi = J[\varphi] \cdot \varphi - W_k[J[\varphi]],$$

which obeys the flow equation (Wetterich 1993, Ellwanger 1994, Morris 1994)

$$\partial_k \Gamma_k[\Phi] = \frac{1}{2}\mathrm{STr}\frac{\partial_k R_k}{\delta^2\Gamma_k[\Phi]/\delta\Phi\delta\Phi + R_k} =$$



Properties of Γ_k: $\Gamma_k \to S$ as $R_k \to \infty$, and $\Gamma_k \to \Gamma$ as $R_k \to 0$.

Important!

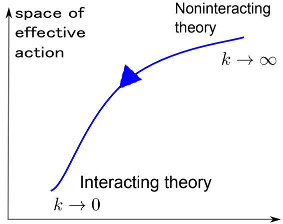
Functional implementation of "block spin transformations" keeps all the information of microscopic systems.

Generalized flow equation of FRG

$\delta S_k[\Phi]$: Some function of Φ with a parameter k. (IR regulator)

k-dependent Schwinger functional

$$\exp(W_k[J]) = \int \mathcal{D}\Phi \exp\left[-\left(S[\Phi] + \delta S_k[\Phi]\right) + J\cdot\Phi\right]$$



Flow equation

$$\begin{aligned}-\partial_k W_k[J] &= \langle \partial_k \delta S_k[\Phi]\rangle_J \\ &= \exp\left(-W_k[J]\right)\partial_k(\delta S_k)\left[\delta/\delta J\right]\exp(W_k[J])\end{aligned}$$

Consequence

We get a (functional) differential equation instead of a (functional) integration!

Application of FRG to a system of composite particles

Model

Classical action:

$$S[\overline{\psi},\psi]=\int_0^\beta \mathrm{d}\tau\int \mathrm{d}^3\boldsymbol{x}\left[\overline{\psi}\left(i\partial_\tau-\frac{\nabla_x^2}{2m}-\mu\right)\psi+g\overline{\psi}_1\overline{\psi}_2\psi_2\psi_1\right].$$

- ψ, $\overline{\psi}$: two-component Grassmannian odd variables dependent on $(\tau,\boldsymbol{x})\in S^1\times\mathbb{R}^3$.
- Thermal boundary condition: $\psi(\tau+\beta,\boldsymbol{x})=-\psi(\tau,\boldsymbol{x})$
- m, μ, and g denote the mass, chemical potential, and coupling constant, resp.

The effective action Γ_k can be defined in the same way with the expansion

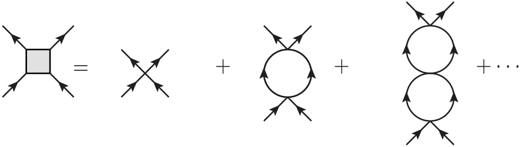
$$\begin{aligned}\Gamma_k[\overline{\psi},\psi] &= \int_p \overline{\psi}_p(-\Sigma_k(p))\psi_p \\ &+ \int_{p,q,q'}\Gamma_k^{(4)}(p;q,q')\overline{\psi}_{1,p/2+q}\overline{\psi}_{2,p/2+q'}\psi_{2,p/2-q'}\psi_{1,p/2-q}.\end{aligned}$$

Renormalization

Most diagrams of this model does not contain UV divergences: (Example)

$$\int \mathrm{d}^3\boldsymbol{p}\frac{1}{\beta}\sum_n \frac{1}{i(2n+1)/\beta+\frac{\boldsymbol{p}^2}{2m}-\mu}=\int \mathrm{d}^3\boldsymbol{p}\frac{1}{\exp\beta(\frac{\boldsymbol{p}^2}{2m}-\mu)+1}<\infty.$$

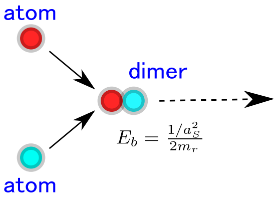
Only exception is the coupling renormalization:



$$\frac{m}{4\pi a_s}=\lim_{\Lambda\to\infty}\left(\frac{1}{g_\Lambda}+\int_0^\Lambda \frac{\mathrm{d}^3\boldsymbol{\ell}}{(2\pi)^3}\frac{1}{2\boldsymbol{\ell}^2/2m}\right)$$

System of composite particles

Positive scattering length : $(k_F a_s)^{-1} \gg 1$
$\Rightarrow$ Low-energy excitations are one-particle excitations of composite dimers.



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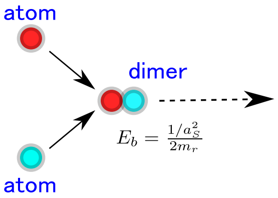


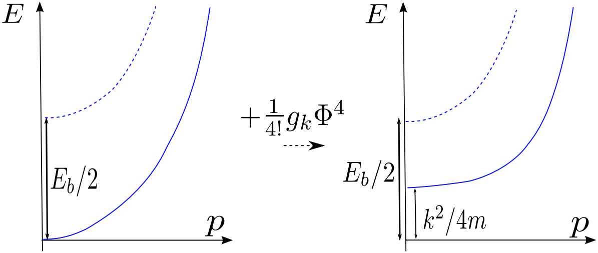
Several approaches for describing BEC of composite bosons. (Pros/Cons)

- Auxiliary field method
  (Easy treatment within MFA/ Fierz ambiguity in their introduction)
- Fermionic FRG ($\Leftarrow$ We develop this method!)
  (Unbiased and unambiguous/ Nonperturbative treatment is necessary)

Vertex IR regulator

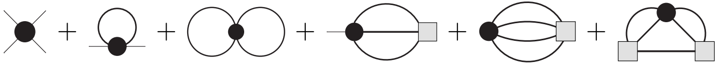
Quartic terms as δS_k:

Vertex IR regulator: $\delta S_k = \frac{1}{4!} g_k^{\alpha\beta\gamma\delta} \Phi_\alpha \Phi_\beta \Phi_\gamma \Phi_\delta$.



Flow equation with the vertex IR regulator (YT, PTEP**2014**, 023A04)

$\partial_k \Gamma_k[\Phi] =$



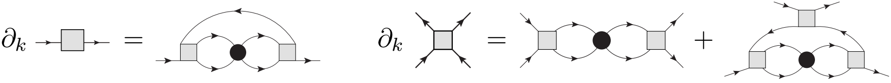
(Blobs: $\partial_k g_k$, Boxes: Field-derivatives of Γ_k)

Vertex IR regulator & Flow equation

Optimization can be satisfied with the vertex IR regulator:

$$\delta S_k = \int_p \frac{g^2 R_k^{(b)}(\boldsymbol{p})}{1 - g R_k^{(b)}(\boldsymbol{p})} \int_{q,q'} \overline{\psi}_{\uparrow,\frac{p}{2}+q} \overline{\psi}_{\downarrow,\frac{p}{2}-q} \psi_{\downarrow,\frac{p}{2}-q'} \psi_{\uparrow,\frac{p}{2}+q'}$$

Flow equation up to fourth order (YT, PTEP**2014** 023A04, YT, arXiv:1402.0283):



Effective boson propagator in the four-point function:

$$\frac{1}{\Gamma_k^{(4)}(p)} = -\frac{m^2 a_s}{8\pi}\left(ip^0 + \frac{\boldsymbol{p}^2}{4m}\right) - R_k^{(b)}(\boldsymbol{p})$$

Flow of fermionic FRG: self-energy

Flow equation of the self-energy:

$$\partial_k \Sigma_k(p) = \int_l \frac{\partial_k \Gamma_k^{(4)}(p+l)}{il^0 + \boldsymbol{l}^2/2m + 1/2ma_s^2 - \Sigma_k(l)}.$$

If $|\Sigma_k(p)| \ll 1/2ma_s^2$,

$$\begin{aligned}\Sigma_k(p) &\simeq \int_l \frac{\Gamma_k^{(4)}(p+l)}{il^0 + \boldsymbol{l}^2/2m + 1/2ma_s^2} \\ &\simeq \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{(8\pi/m^2 a_s) n_B(\boldsymbol{q}^2/4m + \frac{m^2 a_s}{8\pi} R_k^{(b)}(\boldsymbol{q}))}{ip^0 + \frac{\boldsymbol{q}^2}{4m} + \frac{m^2 a_s}{8\pi} R_k^{(b)}(\boldsymbol{q}) - \frac{(\boldsymbol{q}+\boldsymbol{p})^2}{2m} - \frac{1}{2ma_s^2}}.\end{aligned}$$

Estimate of $|\Sigma_k(p)|$:

$$|\Sigma_k(p)| \lesssim \frac{1}{2ma_s^2} \times (\sqrt{2mT}a_s)^3 \times n_B(k^2/4m).$$

⇒ Our approximation is valid up to $(k^2/2m)/T \sim (k_F a_s)^3 \ll 1$.

Critical temperature in the BEC side

Number density:

$$\begin{aligned} n &= \int_p \frac{-2}{ip^0 + \boldsymbol{p}^2/2m + 1/2ma_s^2 - \Sigma_0(p)} \\ &\simeq \frac{(2mT_c)^{3/2}}{\pi^2}\sqrt{\frac{\pi}{2}}\zeta(3/2). \end{aligned}$$

Critical temperature and chemical potential:

$$T_c/\varepsilon_F = 0.218, \qquad \mu/\varepsilon_F = -1/(k_F a_s)^2.$$

⇒ Transition temperature of BEC.

Consequence

FRG with vertex regulator provides a nonperturbative description of many-body composite particles.

Summary

- Renormalization group provides a convenient framework in applications of QFTs.
- Fermionic FRG is a promising formalism for interacting fermions.
  ⇒ Bose–Einstein condensation without explicit bosonic fields.